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A CLASS OF INVERSE CREEP THEORY PROBLEMS

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UDC 539.376

Certain inverse problems associated with finding the external effects needed to obtain the requisite residual body or plate shape under creep conditions in a given time t_* with elastic unloading are taken into account at the time $t = t_*$. It is assumed here that the unknown external effects belong to a definite class, for instance relaxation problems were examined in [2, 4, 5] when unknown displacements of body surface points (unknown plate deflections) remained fixed during the time t_*, and external loads were considered constant in time in [1, 2], etc.

A class of inverse problems about finding external loads such as would assure a given residual body (plate) shape at any running time is investigated in this paper. A theorem on the uniqueness of the solution is proved for the cases of small strains. A variational formulation is given for these problems on the basis of finding the stationary value of a certain functional; the displacement and stress velocities are here varied simultaneously as both running and residual (after elastic unloading). The solution of the problem in an exact formulation is compared in a specific example with the solution obtained by using the mentioned mixed variational principle.

1. Let us consider a uniformly heated body of volume v with surface S whose governing strain equations we write as

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn} + \varepsilon^c_{kl} \qquad (k, l = 1, 2, 3), \tag{1.1}$$

where $\varepsilon_{k\ell}$, $\varepsilon_{k\ell}{}^c$, $\sigma_{k\ell}$, $a_{k\ell mn} = a_{mnk\ell}$ are components of the total strain, creep strain, stress and elastic pliability tensors, respectively, summation from 1-3 is performed over repeated

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 163-173, March-April, 1989. Original article submitted July 12, 1988.

 0021-8944/89/3002-0320$12.50

subscripts. The strains are considered small so that the components $\varepsilon_{k\ell}$ are expressed in terms of the displacement vector components u_k by known Cauchy relationships.

We take the general Rabotnov [7] dependences for the creep strain rates $\eta_{k\ell} = \dot{\varepsilon}_{k\ell}^c$ (the dot denotes differentiation with respect to time t), which yield a good description of the metal creep process and have the following form under isothermal strain:

$$\eta_{kl} = \eta_{kl}(\sigma_{mn},\ q_i)\ \ (k,\ l,\ m,\ n = 1,\ 2,\ 3,\ i = 1,\ 2,\ \ldots,\ r). \tag{1.2}$$

Here q_i is the set of structural parameters whose variation in time is reflected by the kinetic equations

$$\dot{q}_i = \dot{q}_i(\sigma_{mn},\ \varepsilon_{mn}^c,\ q_j) \qquad (i, j = 1, 2, \ldots, r, m, n = 1, 2, 3). \tag{1.3}$$

Let us assume that the medium under consideration satisfies the stability postulate that is formulated as follows for an isothermal creep process [8]: For any two paths in stress space $\sigma_{k\ell}^{(1)} = \sigma_{k\ell}^{(1)}(t)$ and $\sigma_{k\ell}^{(2)} = \sigma_{k\ell}^{(2)}(t)$ and their corresponding paths in creep strain rate space $\eta_{k\ell}^{(1)} = \eta_{k\ell}^{(1)}(t)$ and $\eta_{k\ell}^{(2)} = \eta_{k\ell}^{(2)}(t)$ (k, ℓ = 1, 2, 3), the following inequality is satisfied for any time t > 0

$$\int_0^t \Delta\sigma_{kl}\Delta\eta_{kl}dt \geqslant 0, \tag{1.4}$$

where $\Delta\sigma_{k\ell} = \sigma_{k\ell}^{(1)} - \sigma_{k\ell}^{(2)}$, $\Delta\eta_{k\ell} = \eta_{k\ell}^{(1)} - \eta_{k\ell}^{(2)}$ under the conditions $q_i^{(1)} = q_i^{(2)}$ and $\varepsilon_{k\ell}^{c(1)} = \varepsilon_{k\ell}^{c(2)}$ for t = 0 [in those cases when the right sides of (1.3) do not contain ε_{mn}^c, the condition $\varepsilon_{k\ell}^{c(1)} = \varepsilon_{k\ell}^{c(2)}$ is not certain at t = 0]. It is considered that the equality sign in (1.4) is possible for a medium compressed under creep ($\eta_{kk} \neq 0$), only for $\Delta\sigma_{k\ell}(t) = 0$ (k, ℓ = 1, 2, 3) for any t > 0 and for an uncompressed medium ($\eta_{kk} = 0$), only for $\Delta\sigma_{kl} = \Delta p(t)\delta_{kl}\,(k, l = 1, 2, 3)$ for t > 0 ($\delta_{k\ell}$ are unit tensor components).

Condition (1.4) imposes definite constraints on the creep equations (1.2) and (1.3). These constraints are set up in [8] for fundamental media classes (nonlinearly viscous, hardening, softening, media with strain anisotropies, etc.).

Let us formulate the inverse problem of creep theory: What external loads $p_k = p_k(t)$ must be applied to a body surface S so that it would have given residual displacement $\tilde{u}_k = \tilde{u}_k(t)$ ($\tilde{u}_k = 0$ for t = 0) (k = 1, 2, 3) of the surface points, i.e., those displacements that would remain on S after instantaneous removal of running external loads and elastic unloading? It is considered that the body was in the undeformed state for t < 0 (or, in the more general case, creep strain distributions $\varepsilon_{k\ell}^c$ and values of the parameters q_i are given in the body for t = 0).

Let us prove that if the solution of this problem for a body with the governing equations (1.1)-(1.3) and the constraint (1.4) exists then it will be unique for a body compressed during creep while the external loads p_k for an incompressible body are determined to the accuracy of arbitrary hydrostatic pressure.

We assume for the proof that two solutions exist for this problem that satisfy the identical boundary and initial conditions, we denote their appropriate differences by using the symbol Δ. Since $\Delta\dot{\tilde{u}}_k = 0$ on S for any t > 0, then we obtain from the known equation of virtual work

$$\int_v \Delta\sigma_{kl}\Delta\dot{\tilde{\varepsilon}}_{kl}dv = 0; \tag{1.5}$$

$$\tilde{\varepsilon}_{kl} = a_{klmn}\rho_{mn} + \varepsilon_{kl}^c \qquad (k,\ l = 1,\ 2,\ 3), \tag{1.6}$$

where $\tilde{\varepsilon}_{k\ell}$ and $\rho_{k\ell}$ are the residual strain and residual stress fields and $\rho_{k\ell} = 0$ for t = 0 (when the $\varepsilon_{k\ell}^c$ distribution is given in the body at t = 0 the stresses $\rho_{k\ell}$ are found uniquely, i.e., $\Delta\rho_{k\ell} = 0$ for t = 0).

For any t > 0 we represent the stress field in the form $\sigma_{k\ell} = \sigma_{k\ell}^e + \rho_{k\ell}$ [1, 2] ($\sigma_{k\ell}^e$ are stress components which correspond to the solution of the elastic problem for running values of the external loads). Because of self-equilibration of the stresses $\rho_{k\ell}$ the equality $\int_v a_{klmn}\Delta\dot{\rho}_{mn}\Delta\sigma^e_{kl}dv = 0$ holds [1, 2], which we use with (1.1) to find from (1.5)

$$\int_v (a_{klmn}\Delta\dot{\rho}_{mn}\Delta\rho_{kl} + \Delta\sigma_{kl}\Delta\eta_{kl})\, dv = 0. \tag{1.7}$$

Integrating (1.7) with respect to time between zero and the running time t and taking into account that $\Delta\rho_{k\ell} = 0$ for $t = 0$ we obtain

$$\int_v \left[\frac{1}{2} a_{klmn}\Delta\rho_{mn}(t)\,\Delta\rho_{kl}(t) + \int_0^t \Delta\sigma_{kl}\Delta\eta_{kl}dt\right] dv = 0,$$

which is possible by virtue of the properties of the elastic potential and postulate (1.4) only for $\Delta\rho_{k\ell} = 0$ and $\Delta\sigma_{k\ell} = 0$ (for a compressible body under creep) or for $\Delta\sigma_{k\ell} = \Delta p\delta_{k\ell}$ (for an incompressible body) ($k, \ell = 1, 2, 3$) at any time $t > 0$. It follows from the equilibrium equations that Δp is independent of the coordinates of the body points and can be a function of just the time t. There hence results that, in the first case, the external loads p_k are determined uniquely, while in the second to the accuracy of an arbitrary hydrostatic pressure. The assertion is proved.

Let us note that by using the results obtained in [2], a theorem on uniqueness of the solution of an analogous inverse problem of plate bending under small deflections can be proved; it is here necessary to find the external loads assuring given residual deflections at any time under appropriate boundary conditions.

2. The fundamental equations of the class of problems under investigation include: the governing relationships (1.1)-(1.3) and (1.6); the Cauchy relationships for the running $\varepsilon_{k\ell}$ and residual $\tilde{\varepsilon}_{k\ell}$ strains, the equilibrium equations for running $\sigma_{k\ell}$, and residual $\rho_{k\ell}$ stresses, boundary conditions for the residual stresses $\rho_{k\ell}$, and residual displacements $\tilde{u}_k$, and initial conditions for $\varepsilon_{k\ell}{}^c$ and q_i. Without writing these equations down in the general case, let us examine the problem of pure bending of a rectangular beam of unit width with height h as the simplest example. Find the bending moment $M = M(t)$ that will assure a given residual curvature $\tilde{\varkappa} = \tilde{\varkappa}(t)$. For $t < 0$ the beam was in the undeformed state, i.e., $\varkappa = 0$ for $t = 0$.

We assume the beam material to be subjected to power-law creep. Then, using the standard hypothesis of plane sections [7], we obtain the running and residual strain rates from (1.1) and (1.6) ($-h/2 \le y \le h/2$)

$$\dot{\varepsilon} = \dot{\varkappa}y = \dot{\sigma}/E + B\sigma^n; \tag{2.1}$$

$$\dot{\tilde{\varepsilon}} = \dot{\tilde{\varkappa}}y = \dot{\rho}/E + B\sigma^n, \tag{2.2}$$

where $\varkappa$ and $\tilde{\varkappa}$ are the running and residual curvatures, E is Young's modulus, and B and n are creep constants.

The residual stresses ρ are self-equilibrated, the zeroth bending moment corresponds to them, i.e., for any $t > 0$

$$\int_{-h/2}^{h/2} \rho y dy = 2\int_0^{h/2} \rho y dy = 0. \tag{2.3}$$

Let us determine σ_0, the stress for $t = 0$. It follows from (2.1) that $\sigma_0 = E\varkappa_0 y$. Substituting this value into (2.2) for $t = 0$, multiplying the latter by y and integrating with respect to y between y and h/2 with (2.3) taken into account, we have $\frac{1}{3}\left(\frac{h}{2}\right)^3\dot{\tilde{\varkappa}}_0 = \frac{B(E\varkappa_0)^n}{n+2} \times \left(\frac{h}{2}\right)^{n+2}$, hence $\sigma_0 = \left[\frac{n+2}{3B}\left(\frac{h}{2}\right)^{1-n}\dot{\tilde{\varkappa}}_0\right]^{1/n} y$ ($\dot{\tilde{\varkappa}}_0$ is the rate of residual curvature for $t = 0$).

Let us introduce the dimensionless quantities $\xi = 2y/h$, $\psi = h\varkappa/2$, $\tilde{\psi} = h\tilde{\varkappa}/2$, $\bar{\sigma} = \sigma/p_0$, $\tau = BEp_0^{n-1}t$ ($p_0 > 0$ is the characteristic stress). Then we obtain $\bar{\sigma}_0 = 1/p_0 \times \left[\frac{n+2}{3} \times Ep_0^{n-1}\frac{d\tilde{\psi}}{d\tau}\Big|_{\tau=0}\right]^{1/n}\xi$ for the dimensionless stress for $t = 0$. We select the constant p_0 from the condition $\max\bar{\sigma}_0 = 1$, i.e., $\bar{\sigma}_0 = \xi$ and $p_0 = E\dot{\tilde{\psi}}_0(n + 2)/3$ (here and below, the dot denotes differentiation with respect to the dimensionless time τ). We find for the dimensionless quantities from (2.1)-(2.3) (we omit the bar above the σ and ρ):

$$\frac{3}{n+2}\frac{\dot{\psi}}{\dot{\tilde{\psi}}_0}\xi = \dot{\sigma} + \sigma^n; \tag{2.1'}$$

$$\frac{3}{n+2}\frac{\dot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0}\xi = \dot{\rho} + \sigma^n; \tag{2.2'}$$

$$\int_0^1 \rho\xi\, d\xi = 0. \tag{2.3'}$$

There results from (2.1'), (2.2'), and the initial conditions

$$\sigma(\xi, \tau) = \rho(\xi, \tau) + \alpha(\tau)\xi, \quad \alpha(\tau) = \frac{3}{n+2}\frac{\psi - \tilde{\psi}}{\dot{\tilde{\psi}}_0}. \tag{2.4}$$

Multiplying (2.2') by ξ, integrating with respect to ξ between 0 and 1, and taking account of (2.3'), we have $\frac{1}{n+2}\frac{\dot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0} = \int_0^1 \sigma^n\xi d\xi$. Substituting the value of σ from (2.4) into this relationship, we find

$$\frac{1}{n+2}\frac{\dot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0} = \int_0^1 (\rho + \alpha\xi)^n \xi d\xi, \tag{2.5}$$

and then (2.4) into (2.2') and differentiating (2.5) with respect to τ, we arrive at a system of equations to find the functions $\rho = \rho(\xi, \tau)$ and $\alpha = \alpha(\tau)$:

$$\begin{gathered} \dot{\rho} = \frac{3}{n+2}\frac{\dot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0}\xi - f^n \quad (f = \rho + \alpha\xi), \\ \dot{\alpha} = \left[\frac{1}{n(n+2)}\frac{\ddot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0} - \int_0^1 \dot{\rho} f^{n-1}\xi\, d\xi\right] \Big/ \int_0^1 f^{n-1}\xi^2\, d\xi. \end{gathered} \tag{2.6}$$

It is taken into account in the derivation of the second equation in (2.5) that α is independent of ξ. The initial conditions for system (2.6) have the form

$$\rho(\xi, 0) = 0, \quad \alpha(0) = 1. \tag{2.7}$$

Let us note that (2.5) and the second equality in (2.6) are equivalent by virtue of (2.7).

Numerical integration of system (2.6) with conditions (2.7) causes no difficulties. For instance, for the case when the residual curvature grows at a constant rate, i.e., $\dot{\tilde{\psi}} = \dot{\tilde{\psi}}_0 =$ const, we write (2.6), after simple manipulations, as

$$\begin{gathered} \dot{\rho} = \frac{3}{n+2}\xi - f^n \quad (f = \rho + \alpha\xi), \\ \dot{\alpha} = -\frac{3}{n+2} + \left(\int_0^1 f^{2n-1}\xi\, d\xi\right) \Big/ \int_0^1 f^{n-1}\xi^2\, d\xi. \end{gathered} \tag{2.8}$$

The computation procedure is the following. Segment [0, 1] is partitioned into 21 integration nodes so that the step in ξ would equal 0.05. The first equation of (2.8) is writ-

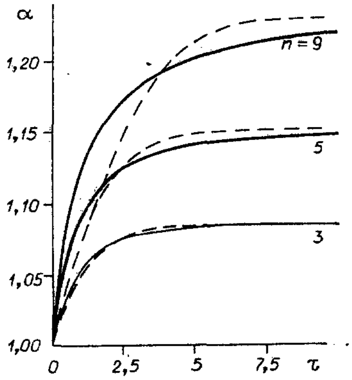

Fig. 1

ten at these nodes, i.e., for different values of ξ. The system obtained in combination with the second equation in (2.8) and the initial conditions is integrated by the Runge–Kutta method. The integrals in (2.8) were here replaced by finite sums of the Simpson formula using the same nodes, i.e., the integration step is $\Delta\xi = 0.05$. The integration step $\Delta\tau$ in the dimensionless time was varied and the results of the computation were compared for different values of $\Delta\tau$; the initial step $\Delta\tau = 0.1$ was selected, which was magnified tenfold after 10 steps. The graphs $\alpha = \alpha(\tau)$ are displayed by continuous lines for n = 3, 5, 9 [α characterizes the bending moment since it follows from (2.3') and (2.4) that $\alpha = 3\int_0^1 \sigma\xi\, d\xi = \frac{6}{p_0 h^2} M$). As is seen from the graphs, α tends to its limit value $\alpha_\infty(n) = \frac{3n}{2n+1}\left(\frac{3}{n+2}\right)^{1/n}$ [$\alpha_\infty(3) = 1.084$, $\alpha_\infty(5) = 1.151$, $\alpha_\infty(9) = 1.230$] which corresponds to the steady-state solution of the system (2.8) as $\tau \to \infty$ when $\dot\rho = \dot\alpha = 0$.

The steady stress distribution has the form

$$\rho_\infty = \left(\frac{3}{n+2}\right)^{1/n}\left(\xi^{1/n} - \frac{3n}{2n+1}\xi\right),\quad \sigma_\infty = \left(\frac{3}{n+2}\right)^{1/n}\xi^{1/n}. \tag{2.9}$$

For instance, for n = 3, the stress distribution for $\tau \geq 5$ practically agrees with (2.9), the corresponding values of τ are somewhat greater for n = 5 and 9, i.e., as n grows the (dimensionless) time of the stress redistribution from the elastic $\sigma_0 = \xi$ to the steady (2.9) also grows.

The running dimensionless curvature ψ is determined from (2.4)

$$\psi/\dot{\tilde\psi}_0 = \tau + (n+2)\alpha/3, \tag{2.10}$$

since $\tilde\psi = \dot{\tilde\psi}_0\tau$ in this case. Therefore, for sufficiently large τ (for example, $\tau \gtrsim 5$ for n = 3) when $\alpha \approx \alpha_\infty$, the running curvature grows at the same rate as the residual $\dot\psi \approx \dot{\tilde\psi}_0$.

3. The class of inverse problems being investigated allows for a variational formulation. Let us examine the functional

$$J = \int_v \left[\dot{\tilde\varepsilon}_{kl}\dot\sigma_{kl} + \dot\varepsilon_{kl}\dot\rho_{kl} - a_{klmn}\dot\rho_{kl}\dot\sigma_{mn} - \eta_{kl}\left(\dot\sigma_{kl} + \dot\rho_{kl}\right)\right] dv - \int_S \dot p_k\left(\dot{\tilde u}_k - \dot{\tilde u}_{k*}\right) dS, \tag{3.1}$$

where $\tilde\varepsilon_{k\ell} = (1/2)(\tilde u_{k,\ell} + \tilde u_{\ell,k})$, $\varepsilon_{k\ell} = (1/2)(u_{k,\ell} + u_{\ell,k})$ are residual and running strain components (the subscript after the comma denotes the partial derivatives with respect to the appropriate Cartesian coordinate); $\eta_{k\ell}$ are creep strain rate components that are determined according to (1.2); p_k are external load components; and $\tilde u_{k*}$ are given residual displacements S (k, ℓ = 1, 2, 3).

Let us vary the functional (3.1) by considering the independent variables $\dot{\sigma}_{k\ell}$, $\dot{\rho}_{k\ell}$, $\dot{u}_k$, and $\dot{\tilde{u}}_k$ ($k, \ell = 1, 2, 3$). Performing the usual calculations in such cases (see [7, pp. 634-637, for instance]) we obtain from (3.1)

$$\delta J = \int_v [(\dot{\tilde{\varepsilon}}_{kl} - a_{klmn}\dot{\rho}_{mn} - \dot{\eta}_{kl})\,\delta\dot{\sigma}_{kl} + (\dot{\varepsilon}_{kl} - a_{klmn}\dot{\sigma}_{mn} - \dot{\eta}_{kl})\,\delta\dot{\rho}_{kl} - \dot{\sigma}_{kl,l}\,\delta\dot{\tilde{u}}_k -$$

$$- \dot{\rho}_{kl,l}\delta\dot{u}_k]\,dv + \int_S [(\dot{\sigma}_{kl}n_l - \dot{p}_k)\,\delta\dot{\tilde{u}}_k + \dot{\rho}_{kl}n_l\delta\dot{u}_k - (\dot{\tilde{u}}_k - \dot{\tilde{u}}_{k*})\,\delta\dot{p}_k]\,dS$$

[n_k ($k = 1, 2, 3$) are components of the unit normal vector external to S].

Because of the independence of the variations $\delta\dot{\sigma}_{k\ell}$, $\delta\dot{\rho}_{k\ell}$, $\delta\dot{u}_k$, and $\delta\dot{\tilde{u}}_k$ in the volume v and $\delta\dot{u}_k$, $\delta\dot{\tilde{u}}_k$, and $\delta\dot{p}_k$ on the surface S there follows from the equality $\delta J = 0$

$$\dot{\tilde{\varepsilon}}_{kl} = a_{klmn}\dot{\rho}_{mn} + \dot{\eta}_{kl},\ \dot{\varepsilon}_{kl} = a_{klmn}\dot{\sigma}_{mn} + \dot{\eta}_{kl},\ \dot{\sigma}_{kl,\,l} = \dot{\rho}_{kl,l} = 0 \text{ in the volume v},$$

$$\dot{\sigma}_{kl}n_l\,\delta\dot{p}_k = \dot{p}_k,\ \dot{\rho}_{kl}n_l = 0,\ \dot{\tilde{u}}_k = \dot{\tilde{u}}_{k*} \text{ on the surface } S\ (k, l = 1, 2, 3).$$

Therefore, the stationarity conditions of functional (3.1) are the physical relations (1.1) and (1.6) written in velocities, including also the elastic unloading equations, the equilibrium equations, and the boundary conditions for running and residual stresses also written in velocities.

The formulated variational principle is analogous to the mixed variational principle for direct creep theory problems [7], except in contrast to this latter, the second set of variables characterizing both the running and the residual stress–strain state (after elastic unloading) is varied in (3.1).

Since the stationarity conditions of the functional (3.1) are equivalent to the problem in velocities, the question arises about the determination of the stress components $\sigma_{k\ell}$ at $t = 0$ (as noted above, the components $\rho_{k\ell}$ are either given uniquely at $t = 0$ or are determined from the given $\varepsilon_{k\ell}{}^c$ distribution at $t = 0$). Let us note that the field $\sigma_{k\ell}$ at $t = 0$ is determined uniquely, as follows directly from (1.7) since $\Delta\rho_{k\ell} = 0$ and $\Delta\sigma_{k\ell}\Delta\eta_{k\ell} \geq 0$ for $t = 0$ because of (1.4).

As is customarily done [7], let us assume that the relations (1.2) are potential, i.e., a function $\Phi = \Phi(\sigma_{k\ell}, q_i)$ exists such that $\eta_{k\ell} = \partial\Phi/\partial\sigma_{k\ell}$ ($k, \ell = 1, 2, 3$). Then to find the initial stresses $\sigma_{k\ell 0}$ it is possible to proceed as follows. We obtain from the virtual work equation and (1.6)

$$\int_S \sigma_{kl0}n_l\dot{\tilde{u}}_{k0}dS = \int_v \sigma_{kl0}\dot{\tilde{\varepsilon}}_{kl0}dv = \int_x \sigma_{kl0}(a_{klmn}\dot{\rho}_{mn0} + \eta_{kl0})\,dv = \int_v \sigma_{kl0}\eta_{kl0}dv,$$

where we consider $\int_v a_{klmn}\dot{\rho}_{mn0}\sigma_{kl0}dv = 0$ [1, 2]. It is easy to see that this equality is more accurate and for such variations $\delta\sigma_{k\ell 0}$ which satisfy the equilibrium equations in accordance with the variations $\delta\varepsilon^e_{k\ell 0} = a_{k\ell mn}\,\delta\sigma_{mn0}$ – the equations in junction with the strain i.e., $\delta\sigma_{k\ell 0}$ should be the variations of some elasticity solution. Then

$$\int_S \delta\sigma_{kl0}n_l\dot{\tilde{u}}_{k0}dS = \int_v \delta\sigma_{kl0}\eta_{kl0}dv = \delta\int_v \Phi(\sigma_{kl0}, q_{i0})\,dv$$

(q_{i0} are given values of the parameters q_i for $t = 0$).

Since the function $\Phi = \Phi(\sigma_{k\ell 0})$ will be convex according to (1.4), it can be shown that among all the elastic stress fields the true initial stress state $\sigma_{k\ell 0}$ yields the minimum of the functional

$$J_0 = \int_v \Phi(\sigma_{kl0})\,dv - \int_S \sigma_{kl0}n_l\dot{\tilde{u}}_{k0}dS \qquad (3.2)$$

($\dot{u}_{k0}$ are given residual displacement velocities of surface points for $t = 0$).

Therefore, the solution of the inverse problem of creep theory under investigation is equivalent to finding the stationary value of functional (3.1), the initial stress fields $\sigma_{k\ell 0}$ here minimize the functional (3.2) under the stipulations made above. This discloses a possibility for constructing approximate solutions of problems of this class.

As an example, let us examine the same problem of beam bending as in Sec. 2. In this case, as is easy to show, functions (3.1) and (3.2) have the form

$$J = \int_0^{h/2} \left[\dot{\tilde{\varepsilon}}\dot{\sigma} + \dot{\varepsilon}\dot{\rho} - \frac{1}{E}\dot{\rho}\dot{\sigma} - B\sigma^n(\dot{\sigma} + \dot{\rho})\right] dy; \qquad (3.1')$$

$$J_0 = \int_0^{h/2} \left(\frac{B}{n+1}\sigma_0^{n+1} - \dot{\tilde{\varkappa}}_0\sigma_0 y\right) dy. \qquad (3.2')$$

By giving an initial stress distribution σ_0 linear in y, that corresponds to the elastic solution, and minimizing (3.2'), we find $\sigma_0 = \left[\frac{n+2}{3B}\left(\frac{h}{2}\right)^{1-n}\dot{\tilde{\varkappa}}_0\right]^{1/n} y$, which agrees with the analogous field in Sec. 2. Using the plane-sections hypothesis and introducing the same dimensionless quantities as in Sec. 2, we obtain from (3.1') that to the accuracy of a constant factor

$$J = \int_0^1 \left\{\frac{\dot{\tilde{\psi}}}{\dot{\tilde{\psi}}_0}\xi\dot{\sigma} + \frac{\dot{\psi}}{\dot{\tilde{\psi}}_0}\xi\dot{\rho} - \frac{n+2}{3}\left[\dot{\rho}\dot{\sigma} + \sigma^n(\dot{\sigma} + \dot{\rho})\right]\right\} d\xi \qquad (3.3)$$

(σ and ρ are dimensionless stresses). It is easy to see that the stationarity conditions (3.3), where the independent variables are $\dot{\psi}$, $\dot{\sigma}$, and $\dot{\rho}$, agree with the relationships (2.1')-(2.3').

As before, let the residual curvature grow at a constant rate, i.e., $\dot{\tilde{\psi}} = \dot{\tilde{\psi}}_0$. Let us give the stress distribution in the form of a combination of elastic ($\sigma = \xi$, $\rho = 0$) and steady (2.9) with the equality (2.4) conserved

$$\sigma = (\alpha + \gamma)\xi + \beta\xi^{1/n}, \ \rho = \gamma\xi + \beta\xi^{1/n}, \qquad (3.4)$$

where α, β, and γ are unknown functions of the time:

$$\alpha(0) = 1, \ \gamma(0) = \beta(0) = 0. \qquad (3.5)$$

Let us note that as $\tau \to \infty$ $\alpha = -\gamma = \frac{3n}{2n+1}\left(\frac{3}{n+2}\right)^{1/n}$, $\beta = \left(\frac{3}{n+2}\right)^{1/n}$, as follows from (2.9).

Substituting (3.4) into (3.3), and taking into account that $\dot{\tilde{\psi}} = \dot{\tilde{\psi}}_0$, we have

$$J = \frac{\dot{\alpha} + \dot{\gamma}}{3} + \frac{n}{2n+1}\dot{\beta} + \frac{\dot{\psi}}{\dot{\tilde{\psi}}_0}\left(\frac{\dot{\gamma}}{3} + \frac{n}{2n+1}\dot{\beta}\right) -$$

$$- \frac{n+2}{3}\left[\frac{\dot{\gamma}(\dot{\alpha} + \dot{\gamma})}{3} + \frac{n}{2n+1}\dot{\beta}(\dot{\alpha} + 2\dot{\gamma}) + \frac{n}{n+2}\dot{\beta}^2\right] -$$

$$- \frac{n+2}{3}\int_0^1 [(\alpha + \gamma)\xi + \beta\xi^{1/n}]^n [(\dot{\alpha} + 2\dot{\gamma})\xi + 2\dot{\beta}\xi^{1/n}] d\xi.$$

Equating the variation δJ calculated under the assumption that $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, and $\dot{\psi}$ are varied independently to zero, i.e., setting $\partial J/\partial\dot{\alpha} = \partial J/\partial\dot{\beta} = \partial J/\partial\dot{\gamma} = \partial J/\partial\dot{\psi} = 0$, we arrive at a system of equations that take the following form after simple manipulations:

$$\frac{\dot{\psi}}{\dot{\psi}_0}=1+\frac{n+2}{3}\dot{\alpha},\ \dot{\gamma}=-\frac{3n}{2n+1}\dot{\beta},\ (n+2)\int_0^1[(\alpha+\gamma)\xi+\beta\xi^{1/n}]^n\xi\,d\xi=1, \tag{3.6}$$

$$\frac{n}{2n+1}-\frac{n(n-1)^2}{3(2n+1)^2}\dot{\beta}-\frac{n+2}{3}\int_0^1[(\alpha+\gamma)\xi+\beta\xi^{1/n}]^n\,\xi^{1/n}\,d\xi=0.$$

According to (3.5), the first of the relationships (3.6) is equivalent to (2.10) and the second to the equality $\gamma=-3n\beta/(2n+1)$ whose substitution into the remaining two relationships of (3.6) while differentiating the third with respect to τ will yield a system of equations in the functions $\alpha=\alpha(\tau)$ and $\beta=\beta(\tau)$:

$$\dot{\beta}=\frac{3(2n+1)^2}{n(n-1)^2}\left(\frac{n}{2n+1}-\frac{n+2}{3}\int_0^1 f_1^n\xi^{1/n}d\xi\right),$$

$$\dot{\alpha}=\dot{\beta}\left[\int_0^1\left(\frac{3n}{2n+1}\xi-\xi^{1/n}\right)f_1^{n-1}\xi\,d\xi\right]\Big/\int_0^1 f_1^{n-1}\xi^2\,d\xi\left(f_1=\alpha\xi+\beta\left(\xi^{1/n}-\frac{3n}{2n+1}\xi\right)\right). \tag{3.7}$$

System (3.7) with the initial conditions (3.5) was integrated by the same method and with the same step $\Delta\tau$ as system (2.8). Graphs of $\alpha=\alpha(\tau)$ for n = 3, 5, 9 are displayed in Fig. 1 by dashed lines. (By virtue of the equality $\gamma=-3n\beta/(2n+1)$ condition (2.3') is satisfied; therefore, α has the same meaning as in Sec. 2, i.e., characterizes the bending moment since $\alpha=3\int_0^1\sigma\xi\,d\xi$.) As seen from the graphs, the difference between the values of α corresponding to the solution of the problem in an exact formulation and those obtained by using the mixed variational principle is small although it increases as n grows. The difference in the diagrams of the stresses σ and ρ is more noticeable. A more rapid stress redistribution from the elastic to the steady (2.9) is characteristic for the solution corresponding to system (3.7). These diagrams are not presented here since we are interested only in α it as precisely characterizes the desired external effect.

4. As has already been noted above, the class of inverse problems under consideration can be extended even to the case of plate bending under creep when the deflections are much less than the plate thickness. In particular, under the ordinary boundary conditions [2], a theorem can be proved on the uniqueness of the solution of the problem of finding the external loads assuring a given residual deflection $\tilde{w}=\tilde{w}(x_1, x_2, t)$, if the governing equations of the plate material have the form (1.1)-(1.3) with the constraint (1.4) and creep strain distributions $\varepsilon_{k\ell}{}^c$ and values of q_i are given for t = 0. The proof duplicates that presented in Sec. 1 by using the virtual work equation obtained in [2].

This same class of problems allows a variational formulation. For instance, it can be shown that the stationarity conditions of the functional (3.1), where the second integral must be omitted while integration in the first is extended over the whole plate volume, will reduce to the following equalities written in the velocities: the relations (1.1) and (1.6), the equilibrium equation for moments $\tilde{M}_{k\ell}$ corresponding to residual stresses ($\dot{\tilde{M}}_{k\ell,k\ell}=0$), and the boundary conditions corresponding to a load-free plate contour after unloading.

The situation is more complicated with the geometrically nonlinear problems when the plate deflections can significantly exceed its thickness. Such problems are encountered in pressure treatment of materials in the creep mode, when we speak about obtaining given residual deflections which indeed govern mainly the residual plate shape [6]. In this case, the uniqueness theorem is already not proved successfully, i.e., it is impossible to restore the external loads uniquely according to the given residual deflections and boundary conditions in general. (A similar situation holds even in direct geometrically nonlinear creep theory problems [7].) Nevertheless, under certain constraints even this inverse problem allows a variational formulation which we examine below.

Let the plate occupy the domain S of the x_1Ox_2 plane bounded by a contour Γ. The problem is to find the external forces needed to obtain the running residual deflection $\tilde{w}=\tilde{w}(x_1, x_2, t)$, $\tilde{w}(x_1, x_2, 0)=0$. Let us assume that these external loads are not large, so that the elastic "spring-back" [2, 6] $w_*=w-\tilde{w}$ is much less than $\tilde{w}$. Then we have for the running and residual strains [2]

$$\varepsilon_{kl} = (1/2)(u_{k,l} + u_{l,k}) + (1/2)(\tilde{w} + w_*)_{,k}(\tilde{w} + w_*)_{,l} - z(\tilde{w} + w_*)_{,kl} \approx (1/2)(u_{k,l} + u_{l,k}) + (1/2)(\tilde{w}_{,k}\tilde{w}_{,l} + \tilde{w}_{,k}w_{*,l} + \tilde{w}_{,l}w_{*,k}) - z(\tilde{w} + w_*)_{,kl}, \quad \tilde{\varepsilon}_{kl} = (1/2)(\tilde{u}_{k,l} + \tilde{u}_{l,k}) + (1/2)\tilde{w}_{,k}\tilde{w}_{,l} - z\tilde{w}_{,kl} \quad (k, l = 1, 2) \tag{4.1}$$

(u_k and $\tilde{u}_k$ are components of the running and residual displacements in the plane of the plate).

Let us note that the assumption $|w_*| \ll |\tilde{w}|$ is not true for small values of t. However, by virtue of the smallness of the external forces, the main contribution to the strain will be given by the bending strains while the nonlinear terms in (4.1) can be neglected for small t, i.e.,

$$(\tilde{w} + w_*)_{,k}(\tilde{w} + w_*)_{,l} \approx \tilde{w}_{,k}\tilde{w}_{,l} + \tilde{w}_{,k}w_{*,l} + \tilde{w}_{,l}w_{*,k} \approx 0.$$

Let us assume that residual displacements $\tilde{u}_{k*}$ are given on the contour Γ. Then we write the functional analogous to (3.1) in the form

$$J = \int_{-h/2}^{h/2} \int_S \left[\dot{\tilde{\varepsilon}}_{kl}\dot{\sigma}_{kl} + \dot{\varepsilon}_{kl}\dot{\rho}_{kl} + \rho_{kl}\dot{\tilde{w}}_{,k}\dot{w}_{*,l} - a_{klmn}\dot{\rho}_{kl}\dot{\sigma}_{mn} - \eta_{kl}\left(\dot{\sigma}_{kl} + \dot{\rho}_{kl}\right)\right]dx_1\,dx_2\,dz - \int_\Gamma \dot{p}_k\left(\dot{\tilde{u}}_k - \dot{\tilde{u}}_{k*}\right)ds, \tag{4.2}$$

where h is the plate thickness, s is the arc length of the contour Γ; $\varepsilon_{k\ell}$ and $\tilde{\varepsilon}_{k\ell}$ are defined in (4.1). Summation over the repeated subscripts in (4.2) is from 1 to 2.

Varied independently in the functional (4.2) are $\dot{\sigma}_{k\ell}$, $\dot{\rho}_{k\ell}$, $\dot{u}_k$, $\dot{\tilde{u}}_k$, and $\dot{w}_*$. Omitting the calculations analogous to those elucidated in [2, 5], we present the final expression

$$\delta J = \int_{-h/2}^{h/2} \int_S \left[\left(\dot{\tilde{\varepsilon}}_{kl} - a_{klmn}\dot{\rho}_{mn} - \eta_{kl}\right)\delta\dot{\sigma}_{kl} + \left(\dot{\varepsilon}_{kl} - a_{klmn}\dot{\sigma}_{mn} - \eta_{kl}\right) \times \delta\dot{\rho}_{kl}\right]dx_1dx_2dz - \int_S \left\{\dot{N}_{kl,l}\delta\dot{\tilde{u}}_k + \dot{\tilde{N}}_{kl,l}\delta\dot{u}_k + [(\tilde{N}_{kl,l}\tilde{w}_{,k})^{\cdot} + (\tilde{N}_{kl}\tilde{w}_{,kl} + \tilde{M}_{kl,kl})^{\cdot}]\,\delta\dot{w}_*\right\}dx_1\,dx_2 + \int_\Gamma \left\{\left(\dot{N}_{kl}n_l - \dot{p}_k\right)\delta\dot{\tilde{u}}_k + \dot{\tilde{N}}_{kl}n_l\delta\dot{u}_k + \left[(\tilde{N}_{kl}n_l\tilde{w}_{,k})^{\cdot} + \dot{\tilde{Q}} + \partial\dot{\tilde{H}}/\partial s\right]\delta\dot{w}_* - \dot{\tilde{G}}\partial\delta\dot{w}_*/\partial n - \left(\dot{\tilde{u}}_k - \dot{\tilde{u}}_{k*}\right)\delta\dot{p}_k\right\}ds.$$

Here $N_{k\ell}$ are membrane forces corresponding to the field $\sigma_{k\ell}$; $\tilde{N}_{k\ell}$ and $\tilde{M}_{k\ell}$ are membrane forces and moments corresponding to the field $\rho_{k\ell}$; $\tilde{Q} = \tilde{M}_{k\ell,\ell}n_k$; $\tilde{H} = \tilde{M}_{k\ell}n_k t_\ell$; $\tilde{G} = \tilde{M}_{k\ell}n_k n_\ell$; n_k, t_k (k = 1, 2) are components of the unit normal and tangent vectors to the contour Γ.

Therefore, the equality $\delta J = 0$ is equivalent to the physical equations (1.1) and (1.6) in velocities, the equilibrium equations for the running and residual quantities

$$\dot{N}_{kl,l} = \dot{\tilde{N}}_{kl,l} = 0 \ \ (k = 1, 2), \ (\tilde{N}_{kl}\tilde{w}_{,kl} + \tilde{M}_{kl,kl})^{\cdot} = 0; \tag{4.3}$$

and the boundary conditions on Γ

$$\dot{N}_{kl}n_l = \dot{p}_k, \quad \dot{\tilde{N}}_{kl}n_l = 0, \quad \dot{\tilde{Q}} + \partial\dot{\tilde{H}}/\partial s = \dot{\tilde{G}} = 0, \quad \dot{\tilde{u}}_k = \dot{\tilde{u}}_{k*} \quad (k = 1, 2). \tag{4.4}$$

The equalities $\tilde{N}_{k\ell,\ell} = 0$ and $\tilde{N}_{k\ell}n_\ell = 0$ on Γ for t = 0 were used in (4.3) and (4.4). In particular, there results from the boundary conditions that the contour Γ is completely free of external loads after unloading.

If the residual displacements $\tilde{u}_k$ are not given on Γ, then the integral over Γ must be omitted in (4.2). In this case the equality $\delta J = 0$ is equivalent to the same relationships (1.1) and (1.6) in velocities and the equilibrium equations (4.3) while the last in the boundary conditions (4.4) must be removed and the first must be replaced by $\dot{N}_{k\ell}n_\ell = 0$.

Finding the initial stress field $\sigma_{k\ell 0}$ reduces, as in Sec. 3, to minimization of a functional of type (3.2) under the condition of existence of the creep potential $\Phi = \Phi(\sigma_{k\ell}, q_i)$. As noted above, we consider here that the loads are small at t = 0 and the bending strains will be fundamental, i.e., strains of the plate middle plane are neglected. Repeating the course of the discussion in Sec. 3, and using the virtual work equation from [2], it is easy to show that among all possible elastic stress fields the true initial stress state $\sigma_{k\ell 0}$ minimizes the functional

$$J_0 = \int_{-h/2}^{h/2} \int_S \Phi(\sigma_{hl0})\, dx_1\, dx_2\, dz - \int_S \dot{\tilde{w}}_0 q_0 dx_1\, dx_2 - \\ - \int_\Gamma \left[(Q_0 + \partial H_0/\partial s)\, \dot{\tilde{w}}_0 - G_0\, \partial \dot{\tilde{w}}_0/\partial n \right] ds, \quad (4.5)$$

where $q_0 = -M_{k\ell 0,k\ell}$ are expressed in terms of Q_0, H_0, and G_0 by the relationships presented above; $M_{k\ell 0}$ are moments corresponding to the field $\sigma_{k\ell 0}$; and $\dot{\tilde{w}}_0$ is the given initial rate of change of the residual deflection. The equalities $\dot{Q} + \partial\dot{H}/\partial s = \dot{G} = 0$ on Γ for $t = 0$ were used in deriving (4.5) and which, in particular, enter into (4.4) and correspond to a load-free contour after unloading.

Therefore, the considered class of inverse creep theory problems about finding the external loads that assure a given residual body or plate shape allow for a variational formulation in both geometrically linear and nonlinear formulations (under additional assumptions about the smallness of the external loads). This is fundamental for the construction of algorithms for the numerical solution of such problems that refer to problems on the pressure treatment of materials in the creep mode.

The author is grateful to I. V. Sukhorukov for calculations performed on the electronic computer.

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